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Engineering Notes

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Optimal Maneuver of a Flexible Arm by Space–Time Finite Element Method

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Introduction

WEIGHT and power limitations imposed on spacecraft and space-based manipulators lead to highly flexible structures. In addition, stringent performance specifications require a high-bandwidth control system that may include a large number of the structure's vibrational modes that may be excited. Achieving performance specifications for this class of systems requires many design iterations on both the structure's configuration and the control law. Traditionally, these tasks were performed sequentially. First the structural design was finalized, and then control system iterations were performed using a fixed structure model as the nominal plant. The tasks of structural modeling and of control design, however, are intimately related and need to be performed concurrently to achieve the maximum possible performance.[1–3] In this paper a new computational approach for modeling and control of a flexible beam is developed. The finite element method is used for both spatial and temporal discretizations[4] and creates the appropriate framework for simultaneous control and structural model design iterations. This approach offers a unified solution strategy to the modeling and control problems. It has the advantage of being a general method (suitable for various problems with no need to find structural model modes) with exponential convergence rate (using spectral elements). The method was developed for a specific system and control objective: designing a point-to-point maneuver of a rigid body with a flexible appendage modeled as Timoshenko beam. The approach, however, can be generalized to flexible systems with arbitrarily distributed actuators along the structure.

Problem Statement

System Model

Consider a flexible appendage of length l cantilevered to a rigid hub of radius R as shown in Fig. 1. Motion is restricted to a horizontal plane, and the system is controlled by a control moment $M(t)$ acting on the hub. For simplicity only small flexural displacements are considered (a model that includes axial displacements can be found in Ref. 5). The equations of motion are given by (see also Ref. 6)

$$\beta\frac{\partial^2 w}{\partial\tau^2} + \beta(\xi + r)\frac{\mathrm{d}^2\alpha}{\mathrm{d}\tau^2} - S\left(\frac{\partial^2 w}{\partial\xi^2} - \frac{\partial\theta}{\partial\xi}\right) = 0$$

$$\frac{\mathrm{d}^2\alpha}{\mathrm{d}\tau^2} + \frac{\partial^2\theta}{\partial\tau^2} - S\left(\frac{\partial w}{\partial\xi} - \theta\right) - \frac{\partial^2\theta}{\partial\xi^2} = 0 \tag{1}$$

$$I_r\frac{\mathrm{d}^2\alpha}{\mathrm{d}\tau^2} + \int_0^1\left\{[\beta(\xi + r)^2 + 1]\frac{\mathrm{d}^2\alpha}{\mathrm{d}\tau^2} + \beta(\xi + r)\frac{\partial^2 w}{\partial\tau^2} + \frac{\partial^2\theta}{\partial\tau^2}\right\}\mathrm{d}\xi = f$$

with the following boundary conditions:

$$w(0,\tau) = 0, \qquad \theta(0,\tau) = 0$$

$$\left.\frac{\partial\theta}{\partial\xi}\right|_{\xi=1} = 0, \qquad \left.S\left(\frac{\partial w}{\partial\xi} - \theta\right)\right|_{\xi=1} = 0 \tag{2}$$

where

$$\xi \triangleq x/l, \quad w \triangleq y/l, \quad \tau \triangleq t\sqrt{E/\rho l^2}, \quad f \triangleq Ml/EI$$
$$\beta \triangleq Al^2/I, \quad r \triangleq R/l, \quad S \triangleq (kG/E)\beta, \quad I_r \triangleq I_m/I\rho l \tag{3}$$

and $\alpha(t)$ is the rigid body rotation, $y(x, t)$ the vertical deflection of the beam with respect to the rigid body motion, ρ the density of the beam, A its area cross section, I the area moment of inertia, $\theta(x, t)$ the bending angle (see Ref. 7 for definition), I_m the moment of inertia of the hub, E Young's modulus of the beam, G shear modulus, and k shear coefficient.[8]

Control Problem Formulation

Given a performance index:

$$J = \int_{\tau=0}^{\tau=\tau^f}\left\{\int_{\xi=0}^{\xi=1}[w(\xi,\tau)Q_w(\xi,\tau)w(\xi,\tau) + \dot{w}(\xi,\tau)Q_{\dot{w}}(\xi,\tau)\dot{w}(\xi,\tau) + \theta(\xi,\tau)Q_\theta(\xi,\tau)\theta(\xi,\tau) + \dot{\theta}(\xi,\tau)Q_{\dot{\theta}}(\xi,\tau)\dot{\theta}(\xi,\tau) + u(\xi,\tau)Q_u(\xi,\tau)u(\xi,\tau)]\,\mathrm{d}\xi + \alpha(\tau)^2Q_\alpha(\tau) + \dot{\alpha}(\tau)^2Q_{\dot{\alpha}}(\tau) + f^2(\tau)Q_f(\tau)\right\}\mathrm{d}\tau \tag{4}$$

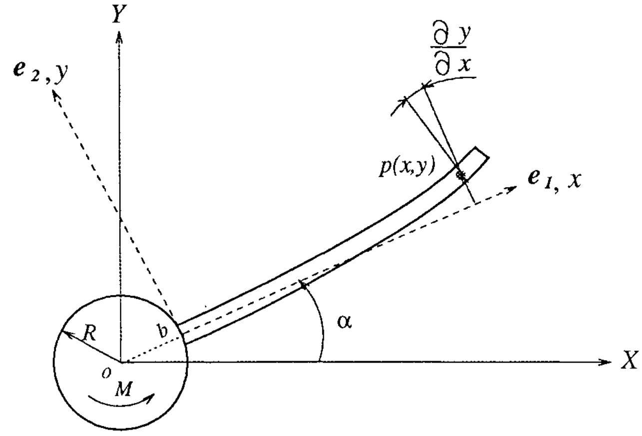

Fig. 1 Dynamic model.

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where the dot denotes derivatives with respect to τ; $Q_w(\xi, \tau) \geq 0$, $Q_{\dot{w}}(\xi, \tau) \geq 0$, $Q_\theta(\xi, \tau) \geq 0$, $Q_{\dot{\theta}}(\xi, \tau) \geq 0$ and $Q_u(\xi, \tau) > 0$ for $0 \leq \xi \leq 1$ and $\tau \geq 0$; $Q_\alpha(\tau) \geq 0$, $Q_{\dot{\alpha}}(\tau) \geq 0$, and $Q_f(\tau) > 0$ for $\tau \geq 0$. The distributed control $u(\xi, \tau)$ is a generalized force along the flexible beam that in our case is identically zero. The objective is to find a control torque $f(\tau)$ that changes the initial orientation of the system to a given orientation in a prescribed time interval while minimizing the performance index J. The performance index can be viewed as a weighted sum of the mechanical energy accumulated in the system during the maneuver and the control effort.

Mathematically, the objective is formulated as follows. Given a set of admissible control functions F find $f(\tau)$ such that

$$\min_{f \in F} J(w, \dot{w}, \theta, \dot{\theta}, \alpha, \dot{\alpha}, f) \tag{5}$$

subject to 1) dynamic equations (1) with boundary conditions (2), 2) initial conditions

$$\begin{aligned} \alpha(0) &= \alpha^0, & \dot{\alpha}(0) &= \dot{\alpha}^0 \\ w(\xi, 0) &= w^0(\xi), & \dot{w}(\xi, 0) &= \dot{w}^0(\xi) \\ \theta(\xi, 0) &= \theta^0(\xi), & \dot{\theta}(\xi, 0) &= \dot{\theta}^0(\xi) \end{aligned} \tag{6}$$

and 3) end conditions

$$\begin{aligned} \alpha(\tau^f) &= \alpha^f, & \dot{\alpha}(\tau^f) &= \dot{\alpha}^f \\ w(\xi, \tau^f) &= w^f(\xi), & \dot{w}(\xi, \tau^f) &= \dot{w}^f(\xi) \\ \theta(\xi, \tau^f) &= \theta^f(\xi), & \dot{\theta}(\xi, \tau^f) &= \dot{\theta}^f(\xi) \end{aligned} \tag{7}$$

Finite Element Formulation

Applying the standard Galerkin method to the equations of motion (1) and employing Green's theorem in both space and time directions, the weak formulation is obtained and discretized by using space–time finite elements.[9] The domain of the solution at each time step contains a set of elements in the space direction and one element in the time direction. The variables of each element are the displacement w and the angles θ and α that are defined within the element and the momenta P_w, P_θ, and P_α that are defined, as follows, on the element temporal boundaries:

$$\begin{aligned} P_w &\triangleq \beta \frac{\partial w}{\partial \tau} + \beta(\xi + r)\frac{d\alpha}{d\tau} \\ P_\theta &\triangleq \frac{d\alpha}{d\tau} + \frac{\partial \theta}{\partial \tau} \\ P_\alpha &\triangleq \int_0^1 \left\{ [\beta(\xi + r)^2 + 1]\frac{d\alpha}{d\tau} \beta(\xi + r)\frac{\partial w}{\partial \tau} + \frac{\partial \theta}{\partial \tau} \right\} d\xi + I_r \frac{d\alpha}{d\tau} \end{aligned} \tag{8}$$

In this formulation C^0 spectral elements are used. The basis functions are Lagrangian interpolants through the Legendre–Gauss–Lobatto points.[10]

For simplicity we use linear polynomials in the temporal direction and polynomials of Nth order in the spatial direction. The dimensionless control torque f is approximated by a constant within the element (this represents an introduction of a zero-order hold in the control loop). Evaluating the appropriate integrals and assembling element matrices in the space direction results in the following representation:

$$\begin{pmatrix} B_{11}, & B_{12} \\ B_{21}, & B_{22} \end{pmatrix} \begin{Bmatrix} q^1 \\ q^2 \end{Bmatrix} = \begin{Bmatrix} \mathcal{P}^1 \\ -\mathcal{P}^2 \end{Bmatrix} + \begin{Bmatrix} d^1 \\ d^2 \end{Bmatrix} \tilde{f}^1 \tag{9}$$

where

$$\begin{Bmatrix} \mathcal{P}^1 \\ -\mathcal{P}^2 \end{Bmatrix} = \begin{pmatrix} C, & 0 \\ 0, & C \end{pmatrix} \begin{Bmatrix} \tilde{P}^1 \\ -\tilde{P}^2 \end{Bmatrix} \tag{10}$$

The matrix B_{ji} is obtained by evaluating the integrals within the element. The vector $q^j = [\tilde{w}^j, \tilde{\theta}^j, \tilde{\alpha}^j]^T$ denotes generalized displacement at time level j, where the variables $\tilde{w}^j$, $\tilde{\theta}^j$, and $\tilde{\alpha}^j$ are the nodal displacement vector, nodal rotation vector, and rigid body rotation, at time level j, respectively. The matrix C is obtained by integrating the spatial base functions on the element temporal boundaries. Using interpolants through the quadrature points yields a naturally diagonalized (lumped) matrix. The vector $\tilde{P}^j = [\tilde{P}^j_w, \tilde{P}^j_\theta, \tilde{P}^j_\alpha]^T$ denotes generalized momentum at time level j, where $\tilde{P}^j_w$ and $\tilde{P}^j_\theta$ are vectors of generalized nodal momentum P_w and P_θ at time level j, respectively. The matrix d^j is obtained by integrating only the temporal base functions. After applying some algebraic manipulations to Eq. (9) it gets the form[9]

$$x^2 = Gx^1 + h\tilde{f}^1 \tag{11}$$

where $x^j = [q^j, \mathcal{P}^j]^T$, G is an amplification matrix, and h has dimensions of a vector.[9]

Solution of the Optimal Control Problem

Using the space–time finite element approach a discretized performance index can be written as

$$\min_{\tilde{f}} J \tag{12}$$

subject to

$$x^{j+1} = Gx^j + h\tilde{f}^j \tag{13}$$

$$x^{n_\tau} = z \tag{14}$$

where

$$J = \sum_{j=1}^{n_\tau - 1} [(x^j)^T A^j x^j + c^j(\tilde{f}^j)^2] \tag{15}$$

and where n_τ is the number of global timewise nodes, A^j a diagonal weighting matrix, and c^j a weighting scalar. Note that the optimization problem defined is over a finite number of parameters (the nodal values of the control torque, generalized displacements, and momenta).

The optimal control effort can be expressed[11] as a feedback control law as follows:

$$\tilde{f}^j = -(k^j)^T x^j - \tilde{f}^j_0 \tag{16}$$

In practice the generalized displacements are measured at finite number of points and estimated at other locations needed for control system implementation.

Results and Discussion

A simulation study was performed to demonstrate the performance of a control law computed using a proposed approach. An open-loop control of a nominal model, i.e., a model used to develop the control law, is presented in Fig. 2. The control effort f, the rigid body angle α, and the end displacement $w(1, \tau)$ are shown. It can be clearly seen that the requirements are satisfied, i.e., the rigid body angle reaches the desired value at the required time with negligible velocity. There exist small high-frequency residual oscillations after completing the maneuver. They could be the result of temporal truncation errors interference. These vibrations can be considerably reduced by applying smaller temporal steps or by implementing a closed-loop control law. The results obtained here are qualitatively similar to the results reported by Breakwell[12] for a similar problem. Breakwell's system model, however, was fixed before the control design was performed; therefore, no structural iterations are possible in parallel with control design. Moreover, control design was done in continuous rather than discrete-time domain.

A phenomenon, known as spillover, is demonstrated in Fig. 3. Here, a comparison of open-loop control of a nominal and of augmented models are shown. The augmented model performance was simulated using a control law computed for a nominal model and then applied to a model with additional degrees of freedom (DOF). As can be observed, open-loop control of the augmented model results in residual oscillations of the unmodeled and, therefore, uncontrolled, high-frequency modes.

In practice only a limited number of modes is excited during a maneuver. Therefore, it is usually possible to design a control law that can inhibit all of the modes excited during the maneuver. Hence,

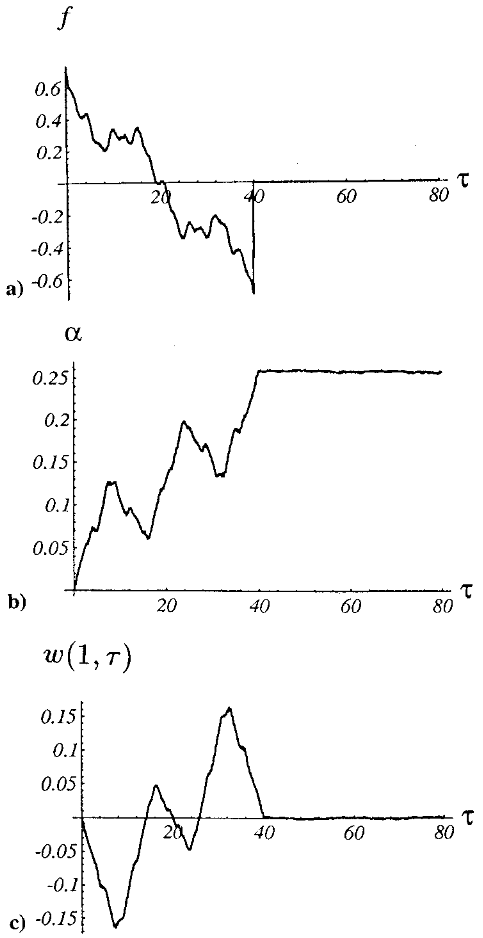

Fig. 2 Open-loop control of a nominal model with $\beta = 1000, S = 0.3\beta, I_r = 0, r = 0.2, \alpha^f = 0.262, N = 4, \Delta\xi = \frac{1}{2}$, and $\Delta\tau/\Delta\xi = 0.05$: a) control law, b) rigid body angle, and c) endpoint displacement.

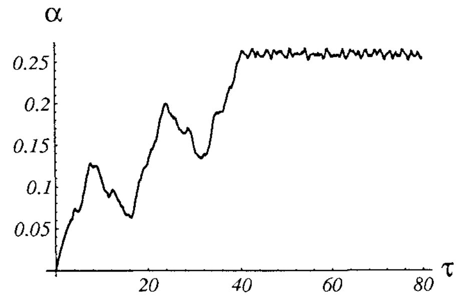

Fig. 3 Influence of higher modes on system performance, case 1, open-loop control of an augmented model with $\beta = 1000$, $S = 0.3\beta$, $I_r = 0$, $r = 0.2$, $\alpha^f = 0.262, N = 6, \Delta\xi = \frac{1}{3}$, and $\Delta\tau/\Delta\xi = 0.02$.

if one includes a sufficient number of DOF in the nominal model, the computed control law will not cause residual oscillations. This is shown in Fig. 4 where open-loop control law computed for a nominal model is applied to an augmented model. The results demonstrate the intimate relationship between structural modeling and control design and the need for developing formulation that allows simultaneous control design and structural model design iterations. It also demonstrates the robustness of the proposed control law with respect to modal truncation, provided that the number of DOF included in the nominal model is large enough. It is expected that in practical applications the performance of control law will be even better since inherent structural damping reduces residual oscillations especially for high-frequency modes. It should be noted, however, that one cannot rely on structural damping alone to reduce residual vibrations. The control law needs to be appropriately designed to reduce high-frequency modal excitation even in presence of modal damping, as is done in the proposed method.

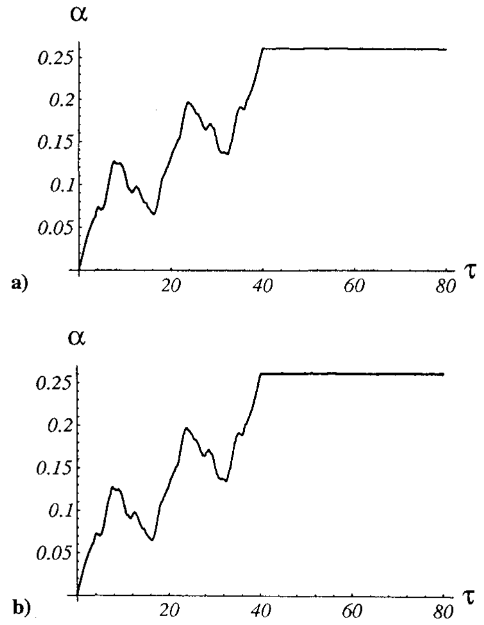

Fig. 4 Influence of higher modes on system performance, case 2, with $\beta = 1000, S = 0.3\beta, I_r = 0, r = 0.2$, and $\alpha^f = 0.262$: a) open-loop control of a nominal model, $N = 6$, $\Delta\xi = \frac{1}{3}$, and $\Delta\tau/\Delta\xi = 0.02$ and b) open-loop control of an augmented model, $N = 10$, $\Delta\xi = \frac{1}{5}$, and $\Delta\tau/\Delta\xi = 0.008$.

Conclusions

The simulation study demonstrated the strong interrelation between structural modeling and control design. It was also shown that with a proper discretization, the proposed control law is robust with respect to modal truncation. Finally, the study demonstrated that imposing equality constraints on the final state yields a very good performance of the nominal system but reduces performance robustness with respect to parameter variations.

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Determination of Weighting Matrices of a Linear Quadratic Regulator

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Introduction

IN optimal control with a quadratic cost function, the selection of the state weighting matrix Q and the control weighting matrix R is normally based on an iterative procedure using experience and physical understanding of the problems involved. To find suitable Q and R that provide a desired balance between the state variable responses and control efforts while satisfying performance requirements and constraints, the transient response of a closed-loop system is typically examined. Because of indirect and nonlinear mapping between the weighting matrices and system closed-loop eigenvalues, it is difficult to find suitable Q and R. Certain general guidelines[1,2] are normally followed to construct Q and R; but these methods may not lead to satisfactory responses. The pole assignment methods[3,4] can provide certain types of connections between closed-loop poles (or eigenvalues) and feedback gains, and it tends to result in more accurate transient responses. By using pole assignment only without consideration of an optimal control cost function, however, it is difficult to balance state and control variables and to account for control effectiveness. The purpose of this Note is to present a systematic method for determining the weighting matrices Q and R to produce specified closed-loop eigenvalues. The method will be demonstrated through a numerical example.

Formulation

A linear system can be described by a state space equation

$$\dot{x} = Ax + Bu \tag{1}$$

where A is the state matrix of $n \times n$, and B is the control matrix of $n \times m$. Also, the pair (A, B) is assumed to be such that the system is controllable. In a linear quadratic regulator (LQR) problem, a cost function described by Eq. (2) needs to be optimized:

$$J = \int_0^\infty (x^T Qx + u^T Ru)\,dt \tag{2}$$

where Q is a positive semidefinite state weighting matrix and R is a positive definite control weighting matrix. The problem can be formulated in terms of a Hamiltonian defined as

$$H = x^T Qx + u^T Ru + \lambda^T(Ax + Bu) \tag{3}$$

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where λ is the vector of Lagrangian multipliers. Then the solution can be obtained by solving the following equations:

$$\dot{\lambda} = -\frac{\partial H}{\partial x} = -A^T\lambda - Qx, \qquad \lambda(\infty) = 0$$
$$\dot{x} = \frac{\partial H}{\partial \lambda} = Ax + Bu, \qquad x(0) = x_0 \tag{4}$$
$$\frac{\partial H}{\partial u} = Ru + B^T\lambda = 0$$

which can be written as

$$\begin{bmatrix} \dot{x} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix} = \tilde{A} \begin{bmatrix} x \\ \lambda \end{bmatrix} \tag{5}$$

where $\tilde{A}$ is a $2n \times 2n$ matrix, and n of its $2n$ eigenvalues are also the eigenvalues of the closed-loop system that satisfy

$$\det[\sigma I - \tilde{A}] = 0 \tag{6}$$

where σ represents the eigenvalues of $\tilde{A}$.

Instead of using an iterative procedure, the following method is proposed to determine the weighting matrices. The weighting matrix R will be chosen to have a diagonal form[1] having elements given by

$$R_{ii} = 1/u_{i,\max}^2, \qquad R_{ij} = 0, \quad i \neq j \tag{7}$$

to penalize each of the control input $u_1, u_2, \ldots, u_m$, and $u_{1,\max}$, $u_{2,\max}, \ldots, u_{m,\max}$ represent the maximum limits of each control input, respectively. The control weighting matrix cofficients can be collectively multiplied by a positive factor without altering the ratios between these coefficients. The weighting matrix Q is also assumed to have a diagonal form with elements given by

$$Q_{ii} = q_i, \qquad Q_{ij} = 0, \quad i \neq j \tag{8}$$

which are to be determined.

Equation (6) will be used to determine n elements $q_i (i = 1, 2, \ldots, n)$ of the weighting matrix Q if all closed-loop eigenvalues are specified. For a specified eigenvalue, $\sigma = \mu + i\omega$, Eq. (6) provides one equation for q_i

$$f(q_1, q_2, \ldots, q_n) = \det[(\mu + i\omega)I - \tilde{A}] = 0 \tag{9}$$

As a result, n algebraic equations,

$$F(s) = [f_1(s), f_2(s), \ldots, f_n(s)]^T = 0 \tag{10}$$

can be solved for the unknown vector $s = (q_1, q_2, \ldots, q_n)$. Expansion of high-order determinants, however, is a tedious task. A computer program[5] based on Chió's algorithm is employed as an alternative to evaluate the $2n$th-order determinant. Newton's method is then used to solve Eq. (10) for the vector s

$$s^{k+1} = s^k - \left[\frac{\partial F}{\partial s}\right]^{-1} \cdot F(s^k) \tag{11}$$

in which the Jacobian matrix, $\partial F/\partial s$ is calculated numerically at s^k. Since the matrix Q is supposed to be positive semidefinite, a negative q_i computed by the given method is replaced by zero, which means no penalty on the corresponding state variable. If several q_i are calculated to be negative, the control weighting matrix will need to be adjusted based on a specific control objective since the required maximum magnitudes of each control input for different objectives are different.

With the resulting weighting matrices Q and R, Ricatti's equation

$$-PA - A^TP + PBR^{-1}B^TP - Q = 0 \tag{12}$$

is used to find matrix P and, hence, the optimal feedback control

$$u = -R^{-1}B^TPx \tag{13}$$